Argumentos
Revista de Filosofia

VARIA

Peter Simons*

# Wittgenstein and surprise in mathematics

"In great mathematics there is a very high degree of unexpectedness".
*G. H. Hardy*

"[T]he mathematician is not a discoverer: he is an inventor."
*Wittgenstein*


## ABSTRACT

One of the psychologically strongest motivations for mathematical platonism is the existence of surprises in mathematics. Time and again results have turned up which went contrary to the expectations of even the best qualified. Wittgenstein was always an anti-platonist, so for him there could be no surprising discoveries about mathematical objects as there can be about animals in the Amazon basin or chemicals on Titan. Given the later Wittgenstein's algorithmic conception of mathematics, it might appear that for him the only legitimate notion of surprise in mathematics must be merely psychological. In this paper I examine whether a less subjective conception is compatible with his position in the philosophy of mathematics.


**Keywords:** Wittgenstein; mathematics; platonism; surprise.


## RESUMO

Uma das mais fortes motivações, em termos psicológicos, para o platonismo matemático é a existência de surpresas em matemáticas. Com frequência, resultados apareceram que foram contrários a expectativas de até mesmo os mais qualificados. Wittgenstein sempre foi um anti-platonista, então para ele não pode existir descobertas surpreendentes sobre objetos matemáticos como pode haver sobre animais na bacia amazônica ou sobre produtos químicos em Titan. Partindo-se da concepção algorítmica do Wittgenstein tardio, deve parecer para que a única noção legítima de surpresa na matemática deve ser uma meramente psicológica. Neste artigo, eu examino se uma concepção menos subjetiva pode ser compatível com a sua posição em filosofia da matemática.


**Palavras-chave:** Wittgenstein; matemática; platonismo; surpresa.

* Trinity College Dublin. Email: psimons@tcd.ie

## Compulsion and surprise

Two phenomena conspire to convince people that the physical world exists independently of them. One is its recalcitrance, or insusceptibility to control. It resists and constrains our actions. Much as we might wish to do so, we cannot lift heavy boulders, walk through walls, jump rivers, breathe under water, or fly (unaided) over mountains. The other feature, which is connected to the first, is the world's propensity to surprise us. The sights and sound, pressures and pains of the world force themselves upon us in perception whether we want them to or not, and are often unexpected and surprising. An unusual bird appears in the garden, a stranger calls at the door and reveals he is a long-lost cousin, the post brings an invitation out of the blue, the car won't start (surprises may be unpleasant as well as pleasant). These two phenomena, recalcitrance and surprise, form a large part of the platonist's case for the existence of an independent mathematical reality. The recalcitrance of mathematical reality indeed appears to be stronger than that of the physical: the necessity with which mathematical results follow from assumptions is stricter than the physical necessity by which a wall resists attempts to walk through it. This has rarely been put more eloquently than by the Polish logician Jan Łukasiewicz. Speaking in particular of mathematical logic, he wrote

> whenever I work on even the least significant […] problem, […] I always have the impression that I am facing a powerful, most coherent and most resistant structure. I sense that structure as if it were a concrete, tangible object, made of the hardest metal, a hundred times stronger than steel and concrete. I cannot change anything in it; I do not create anything of my own will, but by strenuous work I discover in it ever new details and arrive at unshakeable and eternal truths (LUKASIEWICZ, 1970, 249).

One of the most difficult tasks for an anti-platonist, such as Wittgenstein, is to explain this sense of confronting a recalcitrant independently existing reality. And to the end of accounting for the appearance of mathematical necessity and giving it a non-platonist explanation, Wittgenstein devoted much attention.

The phenomenon of surprise in mathematics is also frequently cited as evidence for the independence of mathematical existence. Though it is less widely discussed than the notion of compulsion in mathematics, its persuasive power is if anything greater than that of necessity. Mathematical necessity is pervasive, and mathematicians and commentators on the subject are so used to it that it takes an apparent exception to it to grab their attention. Such an exception is afforded by such results as the independence of the continuum hypothesis, or earlier, the consistency of non-Euclidean geometries. Such exceptions are instances of mathematical surprise, and there are others in the history of mathematics. The most famous is the discovery of incommensurable numbers, a surprise which may even have cost the discoverer his life.[1] Others, more directly relevant to Wittgenstein's intellectual *milieu*, were the paradoxes of set theory, and the incompleteness of arithmetic discovered by Gödel. Because surprises are salient, they provide dramatic phenomenological evidence for the mind-independence of the mathematical. It is therefore incumbent on an anti-platonist like Wittgenstein to find an alternative explanation for the phenomenon of surprise in mathematics, one which rejects the idea that mathematical objects are out there waiting to be discovered, like so many unvisited planets.

## Surprise and the surprising

Whether or not the most penetrating, certainly the most charming philosophical account of surprise that I know is provided by a section of an early work by Adam Smith, a history of astronomy published only posthumously in 1795. Distinguishing surprise from wonder at the novel and admiration of the great, Smith remarks that it is the unexpectedness of what is discovered that constitutes its peculiar feature:

> When an object of any kind, which has been for some time expected and foreseen, presents itself, whatever be the emotion which it is by nature fitted to excite, the mind must have been prepared for it, and must even in some measure have conceived it before-hand; because the idea of the object having been so long present to it, must have before-hand excited some degree of the same emotion which the object itself would excite: the change, therefore, which its presence produces comes thus to be less considerable, and the emotion or passion which it excites glides gradually and easily into the heart, without violence, pain, or difficulty.
> But the contrary of all this happens when the object is unexpected; the passion is then poured in all at once upon the heart, which is thrown, if it is a strong passion, into the most violent and convulsive emotions, such as sometimes cause immediate death; sometimes, by the suddenness of the extacy, so entirely disjoint the whole frame of the imagination, that it never after returns to its former tone and composure, but falls either into a frenzy or habitual lunacy; and such as almost always occasion a momentary loss of reason, or of that attention to other things which our situation or our duty requires. (SMITH, 1967, p. 32).

Smith then goes on to show with graphic examples how dramatic the effects of surprise can be. As this shows, surprise is a psychological reaction to the unexpected, which in intensity may range from mild to overwhelming, indeed sometimes so overwhelming as to prompt disbelief in the supposed datum.

[1] The story is that Hippasus of Metapontum inadvisedly made or announced his discovery while at sea, and was thereupon thrown overboard by scandalized fanatical Pythagoreans.

Surprise is to be distinguished from being surpris*ing*. Something which is usually not itself mental is surprising if it surprises the first people who come upon it or discover it, or which typically surprises those who come across it for the first time in their own experience even after it has become known. It was a surprising discovery that life teems around deep-sea vents: no one, not even experts, had expected there to be life, let alone an abundance of life, in the pitch black of the ocean deep. Something which typically surprises those who experience it for the first time is the size of St. Peter's basilica in Rome. No matter how much they have seen pictures of it, the scale when one is present in person is greater than one would expect. Both of these examples depend on the prior and independent existence of the object in question. So if there is anything in mathematics which is surprising in either sense, if the analogy between the physically surprising and the mathematically surprising holds, it is evidence for the mind-independent existence of mathematical objects.

## Wittgenstein on surprise

At no point in his philosophical career was Wittgenstein prepared to endorse the platonist conception of mathematics. The *Tractatus* is brief about mathematics, but since according to it there are no genuine mathematical propositions, the question of what they are about does not arise. At any rate, "there can never be surprises in logic" (6.1251). Wittgenstein does not go on to say whether there can be surprises in mathematics, but given his tendency to treat logic and mathematics on a par in the *Tractatus*, we must assume he would think there cannot be genuine surprises. In any case, in the sense of surprise being the reaction to something unexpected by those who first come upon it, he was wrong about logic, at least second-order logic. The incompleteness results of Gödel were genuinely surprising at the time, even at first to Gödel, and the hints in the *Tractatus* that there could be a mechanical method for deciding which propositions were logically valid were soon shown by Church to be unfounded even for first-order logic. Wittgenstein had genuine misgivings about Gödel's result, and while his sniping at Gödel's proof is not one of his more impressive efforts, his doubts were shared at the time by more technically versed logicians such as Zermelo and Leśniewski.

Wittgenstein dealt with the notion of the surprising in mathematics in a series of remarks, left out of the first edition of *Remarks on the Foundations of Mathematics* (without explanation) and inserted in the revised edition of 1978 (again without explanation or apology). Their juxtaposition (as Appendix II of Part I) with some of his remarks on Gödel (Appendix III thereof) add weight to the idea that the surprising in mathematics was perceived as a challenge to his anti-platonism.

Wittgenstein distinguishes two roles that surprise can play in mathematics:

> "The surprising may play two completely different parts in mathematics".
> "One may see the value of a mathematical train of thought in its bringing to light something that surprises us:—because it is of great interest, of great importance, to see how such and such a kind of representation of it makes a situation surprising, or astonishing, even paradoxical.
> "But different from this is a conception, dominant at the present day, which values the surprising, the astonishing, because it shews the depths to which mathematical investigation penetrates;—as we might measure the value of a telescope by its shewing us things that we'd have had no *inkling* of without this instrument. The mathematician says as it were: "Do you see, this is surely important, this you would never have known without me." As if, by means of these considerations, as by means of a kind of higher experiment, astonishing, nay *the most* astonishing facts were brought to light."
> "But", protests Wittgenstein immediately, "the mathematician is not a discoverer: he is an inventor." (111)

The first role of surprise is a legitimate one, but it is presentational only: by leading up to a result in a certain way it is highlighted as surprising. The unstated implication is that, were the result presented differently, it would not be surprising. No example is given to illustrate how this can occurs, but here is a possible candidate for the sort of thing Wittgenstein must have had in mind. If we approach set theory via the naïve comprehension principle, using examples to illustrate the principle in action – we have the set of all human mothers, the set of all mothers under thirty years old on 1 January 2000, the set of all teaspoons, and so on – with this background, Russell's Paradox comes as a surprise, even, as to Frege, a devastating bolt out of the blue. On the other hand we may prove in a couple of lines by *reductio* that there is no collection $C$ of objects, no relation $R$ on $C$ and no object $a$ of $C$ such that for all $x$ in $C$, $xRa$ if and only if not $xRx$ – all we need to do is to select $x = a$. From *this* elementary and general perspective, Russell's result follows unsurprisingly as a mere instance by setting $C$ to sets and $R$ to $\in$. Russell's famous barber example is just another instance. As Wittgenstein puts it, "If you are surprised, you have not understood it yet." (ibid.) *Post hoc*, the diagonalizing move that Russell makes, following the pattern set by Cantor, is a commonplace in logic and mathematics, to such an extent that we now find it surprising that Frege should *not* have noticed his logic with *Wertverläufe*[2] violated Cantor's proof that there are more subsets of any set than members of it. In this respect, Wittgenstein is surely right to say that once you see how things work, the surprise fades. Residual surprise is evidence of a lack of understanding, appreciation or firm grasp of how the proof works. Wittgenstein gives the example of being

[2] Value ranges, a kind of object associated with functions, extrapolate from the notion of the extensions of concepts (which are a kind of function for Frege) to all functions.

surprised by an unexpected reduction of a complex algebraic expression, and points out that the psychological effect of surprise is perhaps attendant on concentrating too much on the beginning and the end, and not enough on the steps in between. Surprise at a mathematical result cannot have the *mathematics* as its source: "The surprise and the interest [...] come, so to speak, from outside. I mean, one can say 'This mathematical investigation is of great psychological interest' or 'of great physical interest.'" (112)

Here is an example of how easy it is to misplace the source of surprise. In a lottery game, six numbers are selected at random from 49. One week, the draw throws up six consecutive numbers. "That's amazing!", says A. "No it's not!", says B, "those six numbers had just the same chance of coming up as any other six." B is right about this: in a fair lottery, every selection of six numbers is as likely to come up as every other. But A is right to be surprised. Only one in 317,814 combinations of 6 from 49 has six consecutive numbers, so on average such a combination would turn up, at a rate of two draws a week, about once every three thousand years. The surprise then is that it should happen in a short interval when we are taking note, that an event of such low probability should take place in such a short interval, and the source is physical. But B can rightly retort that any such distribution is equally probable, so the source of surprise is also psychological, since a distribution of six consecutive numbers is much more psychologically salient than all the other equiprobable distributions. In neither case does the mathematics *contribute* to the surprise: on the contrary, it helps to *explain* it.

## Is there ontological surprise in pure mathematics?

Note that Wittgenstein's claim that surprise in mathematics always has a source outside the mathematics, in our own epistemic or imaginative limitations, or in something physical, works only for pure mathematics. There is a different issue about surprise, namely surprise at why and how concepts developed for purely mathematical purposes turn out to have unexpected, indeed startling applications in the physical world. For example, prime numbers, Hardy's favourite example of useless pure mathematics, not only serve an important auxiliary role in Gödel's surprising incompleteness theorems, but extending Fermat's Little Theorem about primes the mathematicians Ron Rivest, Adi Shamir, and Leonard Adleman devised the RSA algorithm for encrypting financial transactions on the internet. Complex Hilbert spaces are the formalism of choice for representing quantum mechanics, yet were developed solely for their own sake. Eugene Wigner (rather histrionically) called this "the unreasonable effectiveness of mathematics in natural science" (WIGNER, 1960), and it has been made the basis of Mark Steiner's account of the universe as "user-friendly" (STEINER, 1998). I must stress that while this is an interesting debate in the area of philosophy of science, cosmology, and *applied* mathematics, it has a quite different point from Wittgenstein's: he was concerned only with the ultimate illegitimacy of surprise within pure mathematics itself.

Wittgenstein is concerned to dispel the idea that there is some *mystery* about pure mathematics, or that there is something deep and hidden, which surprises show up. In this I believe he is right. And his point itself is also neither deep nor mysterious. If pure mathematics consists in drawing conclusions from hypotheses by logically valid reasoning, then the reasons why people are surprised lie in their limitations: a proof is too long to keep all its steps in mind, so something from it is lost from an individual's view. Someone with a clear view of the whole proof from beginning to end will see it all as plain, each step following logically from its predecessors. It may be ingenious and wonderful, and the qualities of the author of the proof may inspire admiration and sometimes surprise, but the mathematics itself gives no legitimate ground for surprise:

> The demonstration has a surprising result!'—If you are surprised, then you have not understood it yet. For surprise is not legitimate here, as it is with the issue of an experiment. There—I should like to say—it is permissible to yield to its charm; but not when the surprise comes to you at the end of a chain of inference. For here it is only a sign that unclarity or some misunderstanding still reigns (111).

Limitations of memory or perception or of grasping complex propositions – in general, *epistemic* limitations – mean that a long or complex proof will be difficult to survey even for the adept. A putative derivation which is too long for anyone possibly to come close to appreciating, and which stubbornly resists such understanding, would cause the putative result simply to be set aside as not proven unless there were good evidence from other sources, such as computer testing, which gave other good reasons (not necessarily themselves pure mathematical reasons) for believing the result. Where the steps of a proof are followed one by one or in groups and found to be valid, but the overall structure remains elusive, the proof will be accepted as difficult and efforts made to understand the structure better or find a shorter proof, both of which are kinds of advance occurring many times in the history of mathematics.

If we hypothetically consider a mathematical proposition which followed logically from accepted hypotheses but whose proof could not possibly be made short enough for a finite creature to follow or appreciate or even write a computer program to test, then such a proposition will simply forever be left as undecided. There is no necessity that famous unresolved propositions about infinite domains such as Goldbach's Conjecture or Riemann's Hypothesis should be resolved at some time in the future, even if as a matter of logic they do (or their negations do) follow from the accepted assumptions.

There will probably always be a stream of pure mathematical results which even the most informed find initially surprising, because, as Wittgenstein rightly points out, until someone has worked through the proof and "looked around", they will know only the result and the starting point. But once the proof has been worked through and understood, the result will fall into place.

The limited role of surprise in pure mathematics can then be explained wholly in terms of the epistemic limitations of human beings in general and (even of) mathematicians in particular, in their difficulty in seeing how one proposition follows from others. There is no reason to call the existence of extra-mental mathematical reality into play to account for such surprises. In this, Wittgenstein was surely right, even if he understandably went too far in the other direction in trying to undermine the credentials of such initially surprising and even dismaying results as Gödel incompleteness.

## References

LUKASIEWICZ, J. 'In Defence of Logistic', in his *Selected Works*. Amsterdam: North-Holland, 1970, p. 236–249.

SMITH, A. 'The Principles Which Lead And Direct Philosophical Enquiries: Illustrated By The History Of Astronomy', from his *Essays on Philosophical Subjects*, 1795, in *The Early Writings of Adam Smith*, ed. J. R. Lindgren. New York: Kelley, 1967, p. 22–35.

STEINER, M. *The Applicability of Mathematics as a Philosophical Problem.* Cambridge, Mass: Harvard University Press, 1998.

WIGNER, E. "The Unreasonable Effectiveness of Mathematics in the Natural Sciences", in *Communications in Pure and Applied Mathematics*, v. 13, 1960.

WITTGENSTEIN, L. 1978. *Remarks on the Foundations of Mathematics*. 3rd. Revised Ed. Oxford: Blackwell.